Determine whether the following series converge or diverge. If it converges, does it converge absolutely or conditionally? If possible, find the sum.

(1) $\sum_{n=1}^{\infty} \frac{n^2+1}{n^5 - n^2\sqrt{3}}$ Rational function, try lim. comp. test w/ $b_n = \frac{n^2}{n^5} = \frac{1}{n^3}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^5+n^3}{n^5-n^2\sqrt{3}} = 1$, So, we have convergence. Also, all terms are positive starting with $n=2$, so the series converges absolutely.

The series converges absolutely by the limit comparison test.

(2) $\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+100}$ rational function, try lim. comp w/ $b_n = \frac{n^2}{n^3} = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3-n}{n^3+100} = 1$ So, the series diverges.

The series diverges by the limit comparison test.

(3) $\sum_{n=1}^{\infty} \left(\frac{n^2}{n^2+1}\right)^n$ Applying the root test here gives an inconclusive result since the limit is 1. Since $\left(\frac{n^2}{n^2+1}\right)^n = \left(1-\frac{1}{n^2}\right)^n$, we should check for divergence.

Let $L = \lim_{n\to\infty} \left(\frac{n^2}{n^2+1}\right)^n$. $\ln L = \lim_{n\to\infty} \ln\left(\frac{n^2}{n^2+1}\right)^n = \lim_{n\to\infty} \underset{\infty}{\underset{\downarrow}{n}}\, \underset{0}{\underset{\downarrow}{\ln\left(\frac{n^2}{n^2+1}\right)}} = \lim_{n\to\infty} \frac{\ln\left(\frac{n^2}{n^2+1}\right)}{\frac{1}{n}} \overset{H}{=} \lim_{n\to\infty} \frac{\frac{\cancel{n^2+1}}{n^{\cancel{2}}} \cdot \left(\frac{\cancel{2n^3}+2n-\cancel{2n^3}}{(n^2+1)^{\cancel{2}}}\right)}{\frac{-1}{n^2}}$

$= \lim_{n\to\infty} \frac{\frac{1}{n^3+n}}{\frac{-1}{n^2}} = \lim_{n\to\infty} \frac{-n}{n^2+1} = 0 \Rightarrow L = 1.$

The series diverges by the divergence test.

(4) $\sum_{n=1}^{\infty} \frac{\cos(2n)}{n^2+1}$

$$\sum_{n=1}^{\infty} \left| \frac{\cos(2n)}{n^2+1} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges by p-series test

converges by comparison test

The series converges absolutely by the comparison test.

(5) $\sum_{n=4}^{\infty} \frac{\ln(n)}{n^3}$

$\ln(n) \leq n$ for all n, so:

$$\sum_{n=4}^{\infty} \frac{\ln(n)}{n^3} \leq \sum_{n=4}^{\infty} \frac{n}{n^3} = \sum_{n=4}^{\infty} \frac{1}{n^2}$$

conv. by p-test

converges by comp. test. Moreover, $\left| \frac{\ln(n)}{n^3} \right| = \frac{\ln(n)}{n^3}$ for $n \geq 1$, so the series actually converges absolutely.

The series converges absolutely by the comparison test.

(6) $\sum_{n=1}^{\infty} \left[\frac{5n}{n+3} - \frac{5(n+1)}{n+4} \right]$

telescoping

$f(n) = \frac{5n}{n+3}$

$$\frac{5n}{n+3} - \frac{5n+5}{n+4} = \frac{\cancel{5n^2}+\cancel{20n}-\cancel{5n^2}-\cancel{5n}-\cancel{15n}-15}{(n+3)(n+4)} = \frac{-15}{(n+3)(n+4)}$$

So, all terms are negative, thus if the series converges it converges absolutely since $|a_n| = -a_n$ for all n here.

$$S_n = f(1) - f(n+1) = \frac{5}{4} - \frac{5(n+1)}{n+4}.$$

$$S = \lim_{n \to \infty} \left(\frac{5}{4} - \frac{5(n+1)}{n+4} \right) = \frac{5}{4} - 5 = \frac{-15}{4}$$

The series converges absolutely by the telescoping series with sum $\frac{-15}{4}$.

(7) $\sum_{n=1}^{\infty} \frac{\sin(n^2)}{n^2}$

$$\sum_{n=1}^{\infty} \left|\frac{\sin(n^2)}{n^2}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$ conv. by p-series test.

conv. by comp. test

The series converges absolutely by the comparison test.

(8) $\sum_{n=1}^{\infty} \frac{n^n}{(n^2+1)^n}$ There are n^{th} powers here, so try root test

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n^n}{(n^2+1)^n}\right|} = \lim_{n\to\infty} \frac{n}{n^2+1} = 0$$

The series converges absolutely by the root test.

(9) $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{2n}}$ n^{th} powers again:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(n!)^n}{n^{2n}}\right|} = \lim_{n\to\infty} \frac{n!}{n^2} = \infty$$

The series diverges by the root test.

(10) $\sum_{n=1}^{\infty} \left(\frac{2^{n+1}}{2^n+1}\right)^n$ n^{th} power again

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2^{n+1}}{2^n+1}\right)^n\right|} = \lim_{n\to\infty} \frac{2^{n+1}}{2^n+1} = \lim_{n\to\infty} 2\left(\frac{2^n}{2^n+1}\right) = 2$$

The series diverges by the root test.

(11) $\sum_{n=0}^{\infty} \frac{3+2^n}{\pi^{n+1}} = \sum_{n=0}^{\infty} \frac{3}{\pi^{n+1}} + \sum_{n=0}^{\infty} \frac{2^n}{\pi^{n+1}} = \sum_{n=0}^{\infty} \frac{3}{\pi}\Big(\underbrace{\tfrac{1}{\pi}}_{r}\Big)^n + \sum_{n=0}^{\infty} \frac{1}{\pi}\Big(\underbrace{\tfrac{2}{\pi}}_{r}\Big)^n = (*)$

*Note: $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$

these are both convergent geometric series (and so absolutely convergent) and the sum is

$$(*) = \frac{\frac{3}{\pi}}{1-\frac{1}{\pi}} + \frac{\frac{1}{\pi}}{1-\frac{2}{\pi}} = \frac{3}{\pi-1} + \frac{1}{\pi-2} = \frac{3\pi-6+\pi-1}{(\pi-1)(\pi-2)} = \frac{4\pi-7}{(\pi-1)(\pi-2)}$$

The series converges absolutely by the geometric series.

(12) $\sum_{n=3}^{\infty} \left[\frac{\ln(n+1)}{n+2} - \frac{\ln(n+2)}{n+3}\right]$ Since $\ln(n)<n$, $\frac{\ln(n+1)}{n+2} - \frac{\ln(n+2)}{n+3} > 0$, so if the series converges, it does so absolutely.

Telescoping with $f(n) = \frac{\ln(n+1)}{n+2}$.

$$S_n = f(3) - f(n+1) = \frac{\ln(4)}{5} - \frac{\ln(n+2)}{n+3}. \quad S = \lim_{n\to\infty} S_n = \frac{\ln(4)}{5}$$

series starts at $n=3$.

The series converges absolutely by the telescoping series

with sum $\frac{\ln(4)}{5}$.

(13) $\sum_{n=1}^{\infty} \frac{2^{n+1}}{3(n!)}$ factorials & powers of constant $\Rightarrow$ try ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{\cancel{n+2}}}{\cancel{3}[\cancel{(n+1)!}]}\cdot\frac{\cancel{3}\cancel{(n!)}}{\cancel{2^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{2}{n+1}\right| = 0$$

The series converges absolutely by the ratio test.

(14) $\sum_{n=1}^{\infty} \frac{2^{2n}}{3\cdot 5^{n-1}} = \sum_{n=1}^{\infty} \frac{4^n}{3\cdot 5^{n-1}} = \sum_{n=1}^{\infty} \frac{4}{3}\left(\frac{4}{5}\right)^{n-1} = \frac{4/3}{1-4/5} = \frac{4/3}{1/5} = \frac{20}{3}$

$\frac{4}{5} = r$

The series is an (absolutely) convergent geometric series, with sum $\frac{20}{3}$.

The series converges absolutely by the geometric series.

(15) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \neq 0.$$

The series diverges by the divergence test.

(16) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2} \quad \text{convergent p-series.}$$

The series converges absolutely by the p-series test.

(17) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = (*)$

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\sqrt{n+1}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}} \geq \sum_{n=1}^{\infty}\frac{1}{\sqrt{2n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{2}}\frac{1}{n^{1/2}} \leftarrow \text{divergent p-series}$$

But, $(*)$ is an alternating series with $b_n = \frac{1}{\sqrt{n+1}}$ &

$\lim_{n\to\infty} b_n = 0$ & $\frac{1}{\sqrt{n+2}} = b_{n+1} \leq b_n = \frac{1}{\sqrt{n+1}}$. So it converges conditionally.

The series converges conditionally by the alternating series test & comparison test

(18) $\sum_{n=1}^{\infty} \frac{(n+1)!}{n^2 e^n}$ factorials & powers of constant $\Rightarrow$ ratio test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\overset{n+2}{\cancel{(n+2)!}}}{(n+1)^2 e^{\cancel{n+1}}}\cdot\frac{n^2 \cancel{e^n}}{\cancel{(n+1)!}}\right| = \lim_{n\to\infty}\frac{n^2(n+2)}{e(n^2+1)} = \infty$$

The series diverges by the ratio test.